International Journal of Non-Linear Mechanics 44 (2009) 538 – 544

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Compact waves on planar elastic rods

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ARTICLE INFO

Article history:
Received 22 August 2008
Received in revised form 28 October 2008
Accepted 28 October 2008

Keywords:
Rods
Waves
Solitary waves
Compact solutions

ABSTRACT

Planar Kirchhoff elastic rods with non-linear constitutive relations are shown to admit traveling wave solutions with compact support. The existence of planar compact waves is a general property of all non-linearly elastic intrinsically straight rods, while intrinsically curved rods do not exhibit this type of behavior.

1. Introduction

Solitary waves are an important class of solutions for non-linear partial differential equations (PDEs). They are waves of *permanent form* that are *localized* within a space region, up to an *exponential tail*. Typically, they are computed using the invariance under time and space translations of the given differential equation. This means that they are obtained by first reducing the PDE to an ordinary differential equation and then computing homoclinic solutions of the reduced dynamical system. Solitary waves are impossible in genuine non-linear hyperbolic systems, where a wave of permanent form will turn into a shock wave. The non-linear terms must be regularized, for instance by a dispersion term, to produce permanent and localized wave forms. *Solitons* are classes of solitary waves that can interact with other solitary waves and emerge from the collision unchanged, except for a phase shift. Solitons were first described by Zabusky and Kruskal in 1965 for the Korteweg–de Vries (KdV) equation and since then solitons have become a fundamental paradigm in non-linear physics [1].

Mathematically, typical solitary waves have infinite tails, but in the real world solitary waves must have a finite span. For this reason Rosenau and Hyman [2] generalized the notion of solitary waves and solitons to admit *compact waves* and *compactons*. A solitary wave with compact support beyond which it vanishes identically is a compact wave. By analogy, a compacton is defined as a compact wave that preserves its shape after interacting with another compacton. Whereas solitary waves are smooth solutions of PDEs, compact waves are weak solutions in the sense that they are not analytic functions and typically present discontinuities in some of their derivatives. A compact wave is a mathematical object that has the advantage that it can describe real world permanent form waves on finite domains in a more accurate way than solitary waves with infinite tails.

The compact wave paradigm is the so called $K(m,n)$-KdV equation [2]. Whereas in the classical KdV equation we have a *non-linear* hyperbolic term that interacts with a *linear* dispersive term, in a $K(m,n)$ equation there is a non-linear hyperbolic term that interacts with a *non-linear* dispersive term. The KdV equation is usually obtained by an asymptotic procedure in different fields of application, such as shallow-water waves with weakly non-linear restoring forces, ion-acoustic waves in a plasma, and acoustic waves on a crystal lattice [3–5]. The KdV equation is one of the best known examples of *universal equations*, which are, therefore, both integrable and widely applicable. By contrast, the $K(m,n)$ equations are somewhat artificial, mathematical toys used to introduce compact waves.

The literature on compact waves and compactons is already extensive. However, to the best of our knowledge, Remoissenet et al. [6,7] were the first to relate a compact wave directly to a physical system in a clear and rigorous way. Moreover, they proposed an experiment to observe compact-like kinks. The experimental apparatus consists of a line of pendulums connected by rubber bands instead of linear springs in an experimental apparatus similar to the

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[1] It is with great pleasure that we dedicate this paper to Philippe Boulanger. Among the contributors to this volume, AG has the unique distinction to have been a student of Philippe Boulanger in both his undergraduate and graduate years, a teaching assistant for his Cours de Mécanique, a colleague in the Département de Mathématiques, a member of his Laboratoire de Mécanique, and, many years later, a collaborator.

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doi:10.1016/j.ijnonlinmec.2008.10.007

mechanical analogue of the sine-Gordon equation [8]. In 2006, using a non-linear theory of elasticity with an inherent material characteristic length, Destrade and Saccomandi were able to prove the existence of pulse solitary waves with compact support [9]. Their idea has been also pursued in Rosenau [10], where an extension to diffusion problems is considered. Details of the mathematics underlying the compact-like wave structures may be found in Refs. [11–13].

In this paper, we are interested in solitary waves propagating in Kirchhoff elastic rods [14]. Waves can propagate as flexural or torsional waves in an elastic rod. The existence of such waves has been studied extensively for both general constitutive relations [15] and, in more details, for the case of inextensible, unshearable rods with quadratic strain-density energy and circular cross-sections [16]. In this particular case, analytical solutions for traveling waves can be obtained based on Kirchhoff analogy between the solutions of the rod problem and the solution of the classical symmetric top [17]. However, numerical simulations have revealed that these solitary waves are not solitons [18]. Other integrable solitary waves for completely and partially integrable rods were obtained in [19] for non-circular cross-sections. The aim of the present paper is to investigate the occurrence of compact-like waves in planar elastic rods with or without intrinsic curvature. In the case of planar rods with a quadratic strain-energy density function, the equations determining the shape of the rods are formally equivalent to the equations of the pendulum. Since all solutions of these equations are known in terms of elliptic functions [20], no compact solution can exist. Therefore, we consider higher-order strain-energy density functions. Doing so, we introduce non-linear dispersive terms in the rod equations which allow for the possibility of compact-like waves. Note that in the mechanics of strings, compact waves are not possible since the equations for the dynamics of elastic strings are not dispersive.

2. Preliminaries

2.1. Kirchhoff equations

We consider a homogenous, inextensible, and unshearable rod. The *rod axis* is the geometric curve connecting the centroids of the rod 'cross sections.' We describe the rod axis in parametric form, by a function $\mathbb{R}\times\mathbb{R}\ni(s,t)\mapsto\mathbf{r}(s,t)\in\mathbb{E}^3$, where the parameter s is the arc length, and t stands for time. A local orthonormal frame of reference $(\mathbf{d}_1(s,t),\mathbf{d}_2(s,t),\mathbf{d}_3(s,t))$ (the *director basis*) is defined at every point of the rod axis, where $\mathbf{d}_3$ is the unit tangent vector pointing in the direction of increasing parameter s, and $\mathbf{d}_1,\mathbf{d}_2$ are oriented along principal axes of inertia of the rod cross section. Following Antman [14], we denote the director basis coordinates of a vector using *sans–serif fonts*, e.g. $\mathbf{a}=\mathsf{a}_1\mathbf{d}_1+\mathsf{a}_2\mathbf{d}_2+\mathsf{a}_3\mathbf{d}_3$. The orientation of the director basis is specified by an axial vector, the *twist vector* $\mathbf{u}(s,t)$, through

$$(\mathbf{d}_i)_s=\mathbf{u}\times\mathbf{d}_i,\quad i=1,2,3, \tag{1}$$

where the subscript s denotes the derivative with respect to the arc length. Note that Eq. (1) is a generalization of the Frenet–Serret equations. If there is no twisting of the rod about its axis (i.e. for pure flexure), the tangential component of the twist vector $\mathbf{u}$ is the torsion τ. The magnitude of the projection of $\mathbf{u}$ onto the $(\mathbf{d}_1,\mathbf{d}_2)$ plane is the curvature κ. Both τ and κ are geometric properties of the rod axis. Moreover, under the same assumption of pure flexure, the director basis is fixed with respect to the Frenet basis $(\boldsymbol{\nu},\boldsymbol{\beta},\boldsymbol{\tau})$, where $\boldsymbol{\tau}\equiv\mathbf{d}_3$ is the tangent, $\boldsymbol{\nu}$ is the principal normal that is the unit vector along the derivative of $\boldsymbol{\tau}$, and $\boldsymbol{\beta}:=\boldsymbol{\tau}\times\boldsymbol{\nu}$ is the binormal vector. Since the rod is inextensible, all strains are described by the twist vector $\mathbf{u}$.

Linear and angular momentum conservation equations yield

$$\mathbf{n}_{ss}=\rho A(\mathbf{d}_3)_{tt}, \tag{2}$$

$$\mathbf{m}_s+\mathbf{d}_3\times\mathbf{n}=[\rho I_2\mathbf{d}_1\times(\mathbf{d}_1)_t+\rho I_1\mathbf{d}_2\times(\mathbf{d}_2)_t]_t, \tag{3}$$

where $\mathbf{n}$ is the force, $\mathbf{m}$ is the moment of force (henceforth: *moment*), ρA is the mass per unit length, $\rho I_1,\rho I_2$ are the moments of inertia about the $\mathbf{d}_1$ and $\mathbf{d}_2$ axes, respectively, and the subscript t denotes the derivative with respect to time.

The material (i.e. elastic) properties of the rod enter the dynamic equations through the constitutive relation,

$$\mathbf{m}=W_{\mathbf{u}}(\mathsf{u}_1-\hat{\mathsf{u}}_1,\mathsf{u}_2-\hat{\mathsf{u}}_2,\mathsf{u}_3-\hat{\mathsf{u}}_3), \tag{4}$$

where W is the *strain-energy density function* (we abbreviate its argument to $\mathbf{u}-\hat{\mathbf{u}}$),

$$W_{\mathbf{u}}(\mathbf{u}-\hat{\mathbf{u}}):=\frac{\partial W}{\partial\mathsf{u}_1}(\mathbf{u}-\hat{\mathbf{u}})\mathbf{d}_1+\frac{\partial W}{\partial\mathsf{u}_2}(\mathbf{u}-\hat{\mathbf{u}})\mathbf{d}_2+\frac{\partial W}{\partial\mathsf{u}_3}(\mathbf{u}-\hat{\mathbf{u}})\mathbf{d}_3$$

is its gradient, and $\hat{\mathbf{u}}(s)$ is the *intrinsic twist* of the rod, i.e. the twist vector in the unstressed state. A naturally straight rod has zero intrinsic twist, $\hat{\mathbf{u}}\equiv\mathbf{0}$.

Eqs. (1)–(4) are collectively called the *Kirchhoff equations* for an elastic rod.

2.2. Traveling wave reduction

Denoting the traveling wave variable by $\chi=s-ct$ (c is the *wave speed*) and the corresponding derivatives with primes, the conservation laws (2) and (3) in the traveling wave system become

$$\mathbf{n}''=c^2\rho A\mathbf{d}_3'', \tag{5}$$

$$\mathbf{m}'+\mathbf{d}_3\times\mathbf{n}=c^2(\rho I_1\mathsf{u}_1\mathbf{d}_1+\rho I_2\mathsf{u}_2\mathbf{d}_2+(\rho I_1+\rho I_2)\mathsf{u}_3\mathbf{d}_3)'. \tag{6}$$

System (1), (5), (6), and (4) represents a system of 18 equations for 18 unknowns $(\mathbf{u},\mathbf{n},\mathbf{m},\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3)$. However, all unknown functions can be expressed in terms of $\mathbf{u}$: the force $\mathbf{n}$ through $\mathbf{u}$ and first integrals, the moment $\mathbf{m}$ explicitly via the constitutive relation, and the director basis vectors $(\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3)$ from $\mathbf{u}$ and the boundary conditions. We will, therefore, refer to $\mathbf{u}(\chi)$ as *the solution of the traveling wave system*.

2.3. Equivalent static system

A remarkable property of Kirchhoff equations is that the form of the traveling wave system is formally equivalent to that of a static system ($c=0$), as described by the following proposition.

Proposition 1. $\mathbf{u}(\chi=s-ct)$ *is a traveling wave solution of Kirchhoff equations, i.e. solution of system* (1), (5), (6), and (4), *if and only if* $\mathbf{u}$ *is a solution of the equivalent static system*

$$\tilde{\mathbf{n}}'=\mathbf{0}, \tag{7}$$

$$\tilde{\mathbf{m}}'+\mathbf{d}_3\times\tilde{\mathbf{n}}=\mathbf{0}, \tag{8}$$

$$\tilde{\mathbf{m}}=\tilde{W}_{\mathbf{u}}(\mathbf{u}-\hat{\mathbf{u}}), \tag{9}$$

where

$$\tilde{\mathbf{n}}:=\mathbf{n}-c^2\rho A\mathbf{d}_3=:\mathbf{n}-T_c\mathbf{d}_3, \tag{10}$$

$$\tilde{\mathbf{m}}:=\mathbf{m}-c^2(\rho I_1\mathsf{u}_1\mathbf{d}_1+\rho I_2\mathsf{u}_2\mathbf{d}_2+(\rho I_1+\rho I_2)\mathsf{u}_3\mathbf{d}_3), \tag{11}$$

$$\tilde{W}(\mathbf{x}):=W(\mathbf{x})-\frac{c^2}{2}(\rho I_1\mathsf{x}_1^2+\rho I_2\mathsf{x}_2^2+(\rho I_1+\rho I_2)\mathsf{x}_3^2)=:W(\mathbf{x})-Z_c(\mathbf{x}), \tag{12}$$

are, respectively, the effective force, effective moment, and effective strain-energy density in the equivalent static system.

Note that in the equivalent static system, the force is reduced by a tension T_c, while the strain-energy density is reduced by a term Z_c quadratic in the strains. Both T_c and Z_c are proportional to the square of the wave speed.

The transformation by Z_c is of particular interest as it allows the adjustment of the quadratic term in the equivalent static strain-energy density by choosing a suitable wave speed c. If the strain-energy density W of the original traveling wave system contains quadratic and higher-order terms, this transformation makes it possible to cancel the quadratic terms, and unravel the effect of higher-order terms. In two dimensions, the quadratic term in W can be canceled completely by Z_c (this occurs when c equals the speed of sound for the rod material). However, in three dimensions, a particular condition relating geometric properties of the rod to its elastic properties must be satisfied to cancel completely the quadratic terms.

Proposition 1 justifies us in turning our attention to static systems exclusively. Henceforth, we consider system (1), (7)–(9), and drop the tildes hereafter.

2.4. First integrals for static systems

The static Kirchhoff equations admit different first integrals related to physical properties, namely:

Force. The force Eq. (7) yields a vector constant of motion, which, without loss of generality, we choose in the $\mathbf{e}_3$ direction

$$\mathbf{n} = F\mathbf{e}_3 = \text{const.} \tag{13}$$

Energy. Another first integral represents a local form of energy

$$H := \mathbf{u}\cdot W_{\mathbf{u}}(\mathbf{u}-\hat{\mathbf{u}}) - W(\mathbf{u}-\hat{\mathbf{u}}) + \mathsf{n}_3 = \text{const}, \tag{14}$$

where $\mathsf{n}_3 = F\mathbf{e}_3\cdot\mathbf{d}_3$ is the (generally non-constant) tension, i.e. the $\mathbf{d}_3$ coordinate of the constant force vector. If the strain-energy density W is a homogenous function with degree of homogeneity k, then

$$H = (k-1)W(\mathbf{u}-\hat{\mathbf{u}}) + \hat{\mathbf{u}}\cdot W_{\mathbf{u}}(\mathbf{u}-\hat{\mathbf{u}}) + \mathsf{n}_3. \tag{15}$$

2.5. Equations in the plane

In two dimensions, the static Kirchhoff Eqs. (1), (7)–(9), simplify to a single ordinary differential equation (ODE) as follows. We confine the rod to the (x,z) plane of a fixed laboratory frame of reference (x,y,z), with basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$, by pointing the binormal vector along the (constant) y-axis, $\mathbf{d}_2 := \mathbf{e}_2$. By doing so, we have ensured that the director basis is a continuous function of s. This is generally not true of the Frenet basis, where the normal and binormal vectors discontinuously change direction at inflection points.

The only non-zero component of the twist vector is now in the binormal direction, $\mathbf{u} = \kappa\mathbf{d}_2$, and the strain-energy density is a function of one scalar variable, $W(\mathbf{u}) \equiv w(\kappa)$, $W_{\mathbf{u}} \equiv (\mathrm{d}w/\mathrm{d}\kappa)\mathbf{d}_2$. Note that we denote the binormal component u_2 of $\mathbf{u}$ by the same symbol as the curvature: κ. Strictly speaking, however, but as the director basis is not identical to the Frenet basis, the continuity of $\mathbf{d}_2$ implies that the signed curvature κ must be allowed to take on negative values (it changes sign at inflection points).

The effective strain-energy density of the equivalent static system (12) reduces to

$$\widetilde{w}(\kappa) = w(\kappa) - \frac{c^2\rho I_2}{2}\kappa^2. \tag{16}$$

We do not impose the requirement of non-intersection in the plane. This can be interpreted as a rod with an infinitesimally small cross-section where non-neighboring parts are stacked on top of each other (with no non-local interactions) if their Cartesian coordinates coincide.

There are two possible ODEs describing the static planar Kirchhoff rod: one using the angle, and the other using the curvature as the dependent variable.

Angle formulation. The constant F can be used to eliminate the force from (8), yielding

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\kappa}(\kappa-\hat{\kappa})\right)' = F\sin\theta, \tag{17}$$

where $\theta(s) := \angle(\mathbf{e}_3, \mathbf{d}_3(s))$. Since $\kappa \equiv \theta'$, Eq. (17) is a second-order ODE in θ.

For a quadratic strain-energy density, Eq. (17) is the pendulum equation, with its well-known solutions, none of which has compact support. For all initial value problems of the pendulum equation the solution exists and is unique. Therefore, by contradiction, it is not possible to stitch together parts of different solutions in order to obtain a compact wave.

Even though Eq. (17) is the simplest equation to integrate in the intrinsically straight 2D case, the angle formulation is not easily generalizable to the intrinsically curved 2D case (for the boundary conditions are specified in terms of the curvature) or to three dimensions. We will, therefore, turn our attention to an alternative formulation.

Curvature formulation. Eliminating n_1 from the normal component of the force equation using the binormal component of the moment equation, and using the energy integral (Section 2.4), we have

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\kappa}(\kappa-\hat{\kappa})\right)'' = \kappa\left(H + w(\kappa-\hat{\kappa}) - \kappa\frac{\mathrm{d}w}{\mathrm{d}\kappa}(\kappa-\hat{\kappa})\right). \tag{18}$$

2.6. Compact waves

A *solitary wave* is a solution for which the strains and their derivatives asymptotically vanish on both ends. For Eq. (18), the solitary wave condition is

$$\kappa(s) \to \hat{\kappa}(s), \quad \kappa'(s) \to \hat{\kappa}'(s) \quad \text{as } s \to \pm\infty. \tag{19}$$

A *compact wave* is a solitary wave with compact support $[-\ell,\ell]$, i.e. a wave in which the intrinsic state is reached with a finite value $\pm\ell$ of the independent variable s ('in finite time') rather than approaching it asymptotically (exponentially). By analogy with dynamical systems, it is often easier to think of the variable s as a time and picture the solution as evolving in time rather than space. In this paper, we (ab)use the word 'time' to refer to the independent variable of the reduced dynamical system.

For Eq. (18), the compact wave condition is

$$\kappa(s) = \hat{\kappa}(s), \quad \kappa'(s) = \hat{\kappa}'(s), \quad \forall s \in (-\infty,-\ell]\cup[\ell,\infty), \ \ 0<\ell<\infty. \tag{20}$$

In other words, a compact wave is a solution $\mathbf{u}(s)$ that differs from the intrinsic state $\hat{\mathbf{u}}(s)$ only over a bounded set of values of its argument. We will assume that the rod is infinite, so that a compact wave is composed of three parts: two semi-infinite parts with zero strain bridged by a finite part with non-zero strain. Thus

$$\mathbf{u}(s) = \begin{cases} \hat{\mathbf{u}}(s), & s \in (-\infty,-\ell)\cup(\ell,+\infty), \\ \mathbf{v}(s), & s\in[-\ell,\ell], \end{cases} \tag{21}$$

where $\mathbf{v} \not\equiv \hat{\mathbf{u}}$. At the boundaries between the three regions, continuity of $\mathbf{u}$ is required

$$\mathbf{v}(-\ell) = \hat{\mathbf{u}}(-\ell), \quad \mathbf{v}(\ell) = \hat{\mathbf{u}}(\ell). \tag{22}$$

If the derivative $\mathbf{u}'$ is continuous as well, the solution is a *classical solution*, otherwise it is a *weak solution* of the Kirchhoff equations.

The problem of finding a compact wave is, therefore, a boundary value problem for $\mathbf{v}$ subject to boundary conditions (22), where the length of the interval 2ℓ is left unspecified (ℓ is a parameter to be determined from the equations). The boundary values correspond to points $(\hat{\mathbf{u}}(\pm\ell), 0)$ in phase space, and $\mathbf{v}$ corresponds to an orbit connecting these two points. If the intrinsic twist is constant ($\hat{\mathbf{u}}=$const), the two boundary points coincide, and both $\mathbf{v}$ and $\mathbf{u}$ correspond to a closed orbit starting and ending at $(\hat{\mathbf{u}}, 0)$.

3. Compact waves in two dimensions

3.1. Compact wave criterion

We now examine the conditions under which Eq. (18) admits solutions with compact support. We assume that w is a power function, and that the intrinsic curvature is constant ($\hat{\kappa} =$ const).

The general question is as follows. Given a boundary value problem for an ODE, what are the conditions under which the solution's orbit in phase space is traced in finite time? We only consider homoclinic orbits with the homoclinic point at the origin. We assume that the orbit itself has a finite length, and we consider separately the boundaries and the interior of the orbit. More precisely, the orbit is traced in finite time if and only if:

1. the orbit leaves any neighborhood of the origin in finite time,
2. the orbit spends a finite time in any neighborhood of any point in the interior of the orbit.

We refer to these properties as the *finite-time property at the boundaries*, and the *finite-time property in the interior*, respectively. The necessary and sufficient condition for the former is given by the following lemma.

Lemma 2. *Consider the equation*

$$(z^n)'' = P(z), \tag{23}$$

where $P(z) = \sum_{j=0}^{k} a_j z^j$ is a polynomial, and $n > 1$, subject to the null Dirichlet boundary conditions

$$z(-\ell) = z(\ell) = 0, \tag{24}$$

where ℓ is a parameter to be determined from the equation. Let m_0 be the multiplicity of the root $z=0$ in $P(z)$. Then ℓ is finite (i.e. the solution satisfies the finite-time property at the boundaries) if and only if

$$n \neq m_0 \geqslant n - 2. \tag{25}$$

Proof. Eq. (23) is a potential system in terms of $y := z^n$:

$$y'' = \frac{\mathrm{d}V}{\mathrm{d}y}(y), \tag{26}$$

$$V(y) = -\int_0^y P(\eta^{1/n})\,\mathrm{d}\eta. \tag{27}$$

The Dirichlet boundary conditions (24) imply that both y and $y' = nz^{n-1}z'$ are zero at the boundaries. In order to satisfy these boundary conditions, the energy E of the potential system must be set to zero

$$\frac{1}{2}(y')^2 + V(y) = E = 0. \tag{28}$$

Solving for z' yields

$$z' = \frac{2}{\sqrt{n}}\sqrt{\sum_{j=0}^{k} \frac{a_j}{j+n} z^{j-n+2}}. \tag{29}$$

Expanding the root around $z = 0$, the derivative z' becomes, to first order

$$z' \propto z^{((m_0-n)/2)+1}, \tag{30}$$

where m_0 is the multiplicity of the root $z=0$ in $P(z)$. We first impose the regularity restriction: in order for the orbit to remain bounded, the power of z in (30) must be non-negative

$$m_0 \geqslant n - 2. \tag{31}$$

The critical value of n for which the behavior is exponential—and cannot satisfy the boundary conditions (24)—is $m_0=n$, while all other values ($m_0 \neq n$) satisfying (31) yield solutions that converge to $z = 0$ polynomially, which is consistent with the boundary conditions. □

The finite-time properties in terms of the original equation translate directly into finite-time properties of the potential system because the transformation $y(s) = z(s)^n$ does not depend on the 'time' s explicitly. The finite-time property in the interior is guaranteed if V is a coercive function and has a finite depth, and has no quadratic extrema.

3.2. General power-law strain-energy density

We now apply Lemma 2 to Eq. (18) with a general homogenous strain-energy density,

$$w(x) = \frac{\alpha}{k}x^k, \quad k > 2. \tag{32}$$

Only even values of k are considered, since odd values do not correspond to a stable unstressed state. The variables z in the prototype Eq. (23), y in the corresponding potential system (26), and the curvature κ are related by

$$\kappa - \hat{\kappa} = z = y^{1/n} = y^{1/(k-1)}. \tag{33}$$

The key feature of Eq. (18) in the case of non-linear elasticity ($k > 2$) is its singularity: the left-hand side is

$$\alpha((\kappa - \hat{\kappa})^{k-1})'' = \alpha((k-1)(\kappa - \hat{\kappa})^{k-2}\kappa')'. \tag{34}$$

Near the boundaries, the curvature κ approaches the intrinsic curvature $\hat{\kappa}$, thus the function multiplying the highest derivative in (18) approaches zero and reaches zero for a finite value of the independent variable. The right-hand side of (18) must also equal zero for $\kappa = \hat{\kappa}$, thus

$$\hat{\kappa}H = 0. \tag{35}$$

Therefore, *in the intrinsically curved case* ($\hat{\kappa} \neq 0$), *the energy integral H must be equal to zero*, otherwise the boundary conditions cannot be reached, not even asymptotically.

It is interesting to note that, as we have a singular equation (cf. (34)) that can be cast as a potential system (26), (27), it is the non-linear transformation (33) between the two that carries the singularity.

Potential. Applying Lemma 2 to the curvature Eq. (18) for the power function strain-energy density (32), in the left-hand side we identify $n = k - 1$, while the right-hand side is

$$P(z) = \frac{H}{\alpha}\hat{\kappa} + \frac{H}{\alpha}z - z^{k-1}\left(\hat{\kappa}^2 + z\hat{\kappa}\left(2 - \frac{1}{k}\right) + z^2\left(1 - \frac{1}{k}\right)\right). \tag{36}$$

The potential $V(y) = -\int_0^y P(\eta^{1/(k-1)})\,\mathrm{d}\eta$ derived from (36) is a double-well potential. Condition (35) ensures that the origin is at a local extremum of V.

Note on zero H and intrinsically curved rods. If $H = 0$, meaning that there is no tension in the non-strained boundary parts $(-\infty, -\ell)$, (ℓ, ∞), and that the force is compressive in the strained part $[-\ell, \ell]$ ($\mathsf{n}_3 < 0$, cf. (15)), the origin is a local minimum of $V(y)$ for arbitrary even k. The origin is, therefore, a fixed point, and the only solution is the trivial one: $\kappa \equiv \hat{\kappa}$. Therefore, there are no compact wave

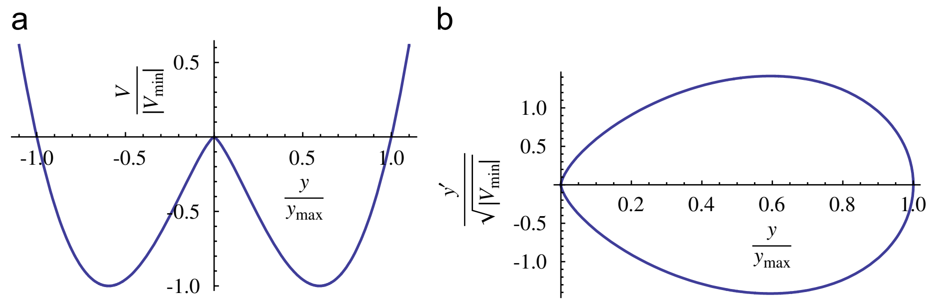

Fig. 1. The potential and the homoclinic orbit for an intrinsically straight rod and a quartic strain-energy density (42). Scales: $y_{max} := 64/a^3$, $V_{min} := -4608/a^8$, where a is the characteristic length scale (46). (a) Potential and (b) zero-energy orbit in (y, y') phase plane.

solutions for $H = 0$, implying that *no compact waves are possible on intrinsically curved rods* $\hat{\kappa} \neq 0$ (cf. (35)). Henceforth, we assume that $H \neq 0$, and consider the intrinsically straight case exclusively

$$H \neq 0, \quad \hat{\kappa} = 0, \tag{37}$$

in which case the origin is at the central local maximum of the double-well potential V.

Behavior near the boundaries. With $\hat{\kappa} = 0$, (36) reduces to

$$P(z) = \frac{H}{\alpha} z - \left(1 - \frac{1}{k}\right) z^{k+1}. \tag{38}$$

The coefficient of the linear term is $(H/\alpha) \neq 0$, yielding $m_0 = 1$, and the criterion (25) is satisfied for all $k > 2$.

Behavior away from the boundaries. For an intrinsically straight rod, the potential is

$$\begin{aligned} V(y) &= -\int_0^y P(\eta^{1/(k-1)})\,\mathrm{d}\eta \\ &= \frac{k-1}{2k^2} y^{k/(k-1)} \left((k-1) y^{k/(k-1)} - 2k \frac{H}{\alpha} \right). \end{aligned} \tag{39}$$

This double-well potential is a coercive function, and has a finite depth. It is straightforward to verify that it satisfies finite-time requirement in the interior for an arbitrary power $k > 2$.

3.3. *Quartic strain-energy density*

A quartic system. Consider a linearly elastic rod, with strain-energy density $w_0(x) = (A/2)x^2$, where A is the flexural rigidity, that is the product of Young's modulus by the cross-section second moment of area. If this strain-energy density is perturbed by a quartic term, $w(x) = w_0(x) + (\alpha/4)x^4$, the quartic term can be brought to light by means of Proposition 1: a wave propagating on this rod with a suitably chosen speed is equivalent to a purely quartic-w static system, the quadratic term being canceled by Z_c (cf. (12)). The critical speed at which this occurs, $c_S = \sqrt{E/\rho}$, is the *speed of sound* in the material of the rod.

Potential system. For the quartic strain-energy density

$$w(x) = \frac{\alpha}{4} x^4, \tag{40}$$

the right-hand side polynomial is

$$P(z) = \frac{H}{\alpha} z - \frac{3}{4} z^5, \tag{41}$$

and applying Lemma 2, we have $m_0 = 1$, $n = 3$, and (25) becomes $3 \neq 1 \geqslant 1$. The potential (39) is

$$V(y) = \frac{3}{8} y^{4/3} \left(\frac{3}{4} y^{4/3} - 2\frac{H}{\alpha} \right), \tag{42}$$

and is shown in Fig. 1. Again, this is a double-well potential with the origin sitting at the local maximum. The homoclinic orbit is shown in Fig. 1.

Integrating the curvature equation. Rather than integrating the potential system, it is more convenient to go back to the curvature equation (18) with the quartic strain-energy density (40),

$$(\alpha\kappa^3)'' = \kappa \left(H - \frac{3\alpha}{4} \kappa^4 \right), \tag{43}$$

which can be solved exactly. Multiplying through by $(\kappa^3)' \equiv 3\kappa^2\kappa'$, the resulting equation can be integrated to yield

$$\frac{\alpha}{2} [(\kappa^3)']^2 = \frac{3H}{4} \kappa^4 - \frac{9\alpha}{32} \kappa^8 + C, \tag{44}$$

where C is an integration constant, which corresponds to the energy in the potential system for y, $C \equiv E$. The only value of the integration constant which yields solutions compatible with the null Dirichlet boundary conditions is $E = 0$. Moreover, any non-zero value of E implies a blow-up of the derivative at the origin, and the only value leading to a finite jump in κ' at the boundaries is $E = 0$. With the integration constant set to zero, dividing both sides of (44) by $(9\alpha/2)\kappa^4$ leads to

$$(\kappa')^2 = \frac{1}{6} \frac{H}{\alpha} - \frac{1}{16} \kappa^4. \tag{45}$$

The solution of this equation can be expressed in terms of the Jacobi elliptic sine function

$$\kappa(s) = \frac{4}{a} \mathrm{sn}(s/a; -1), \quad a := 2\sqrt[4]{\frac{6\alpha}{H}}. \tag{46}$$

The Jacobi sn function is periodic with period $4K(-1)$, where K is the complete elliptic integral of the first kind. In order to construct a compact wave of the form (21) with $\hat{\mathbf{u}} \equiv \mathbf{0}$, we need a half-period between two consecutive zeros. The size of the support of the compact wave solution is thus $2\ell = 2aK(-1) \approx 10.49a$. Therefore, the compact wave solution is given by

$$\kappa(s) = \begin{cases} \dfrac{4}{a} \mathrm{sn}(s/a; -1), & s \in [0, 2aK(-1)], \\ 0 & \text{otherwise}. \end{cases} \tag{47}$$

(The support is now $[0, 2\ell]$, which is equivalent to $[-\ell, \ell]$ up to a phase that is the integration constant in (46), which we have set to zero for simplicity.) The solution (47) is shown in Fig. 2b, and the shape of the rod corresponding to this curvature is shown in Fig. 2c.

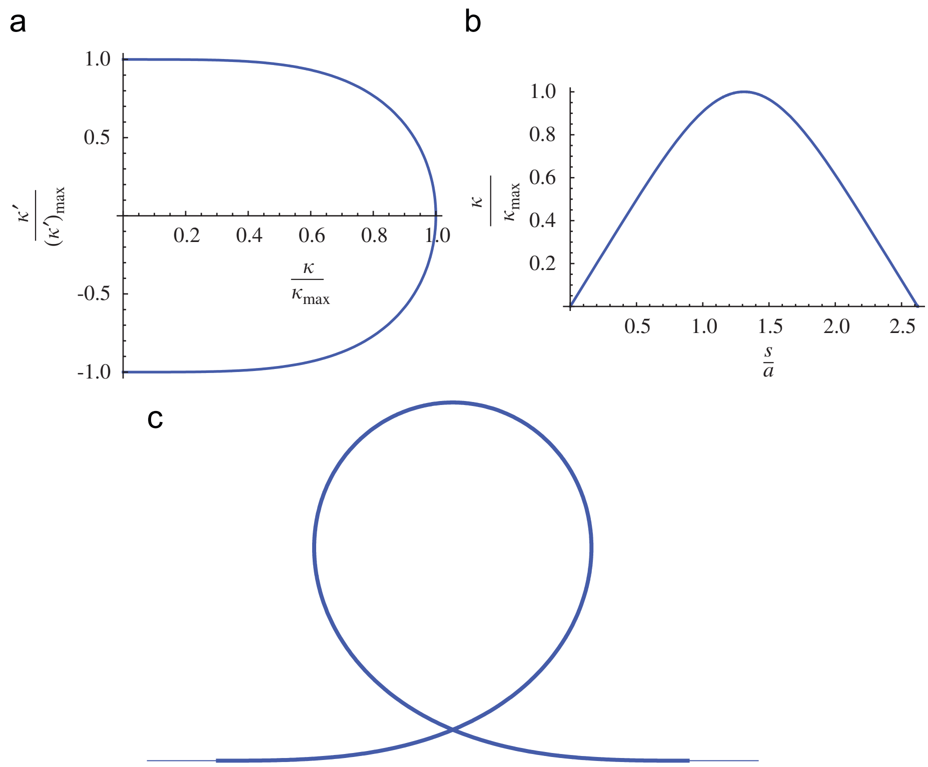

Fig. 2. Compact wave solution on an intrinsically straight rod with quartic strain-energy density (47). Scales: $\kappa_{\max} := 4/a$, $\kappa'_{\max} := 4/a^2$, and a is the characteristic length scale (46). (a) Orbit in (κ, κ') phase plane, (b) curvature as a function of the 'time' parameter s and (c) rod in the plane (the strained part is highlighted).

We have obtained a one-parameter family of compact wave solutions, where the characteristic length a is a function of the ratio of the elastic constant α to the energy integral H.

The following properties of the exact solution (46) follow from the properties of the sn function:

- The graph of sn passes through the origin with unit slope, so (the jump in) the derivative of the curvature (47) at the boundary is $4/a^2$. Solution (47) is a weak solution.
- The sn function reaches a maximum at $s = K(-1)$ with value equal to one. Therefore, the maximum value of the curvature (46) is

$$\kappa_{\max} = \kappa(s = aK(-1)) = \frac{4}{a}. \tag{48}$$

- The derivative of sn can also be expressed in terms of Jacobi elliptic functions

$$\kappa'(s) = \frac{4}{a^2}\,\mathrm{cn}(s/a; -1)\,\mathrm{dn}(s/a; -1); \tag{49}$$

thus the phase portrait can be drawn using exact values (46), (49) (see Fig. 2).

4. Conclusion

Unlike the linearly elastic rod, a static Kirchhoff rod with quartic or higher-order strain-energy density is described by a *singular* ODE (cf. (34)). In a classical homoclinic solution, the fixed point is reached in infinite time. Therefore, there is no possibility of combining the homoclinic solution with any other solution. By contrast, if a fixed point can be reached in a finite time, different solutions can be combined on a bounded time interval. A finite-time homoclinic orbit combines features of an infinite-time homoclinic orbit and a periodic orbit: like an infinite-time homoclinic orbit, it has a fixed point at both ends, and like a periodic orbit, it only takes a finite time for a complete cycle. Having reached the fixed point at either end, based on the non-uniqueness property, the solution can go for another round along the homoclinic orbit (a periodic solution), *or* stay at the fixed point indefinitely (a compact support solution).

We have shown that static compact support solutions always exist on intrinsically straight rods with non-linear constitutive relations, and that intrinsically curved rods do not exhibit this type of behavior.

The results were obtained using homogenous strain-energy densities, but they are also applicable to non-homogenous ones, since the finite-time criterion (25) only depends on the lowest power appearing in the strain-energy density.

Although all results have been reached in a static framework, Proposition 1 provides a broader context: a static solution for a system with strain-energy density W corresponds to a wave traveling at the speed of sound on a rod whose strain-energy density combines W with a quadratic term (cf. (12)). Thus, each static solution with compact support also represents a *compact wave* traveling at the speed of sound. Similarly, compact waves in non-linear Klein–Gordon equations are possible only if the waves move at the speed of sound [6,7].

Acknowledgments

The authors would like to thank John H. Maddocks for stimulating discussions. This material is based in part upon work supported by the National Science Foundation under Grant no. DMS-0604704 (A.G.).

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