International Journal of Non-Linear Mechanics 47 (2012) 197–205

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Non-linear waves in heterogeneous elastic rods via homogenization

Manuel Quezada de Luna [a], Bojan Đuričković [b], Alain Goriely [c,*]

[a] Division of Mathematical and Computer Sciences & Engineering King Abdullah University of Science and Technology, Saudi Arabia
[b] Program in Applied Mathematics, University of Arizona, Tucson, USA
[c] OCCAM, Mathematical Institute, University of Oxford, Oxford, UK

ARTICLE INFO

Available online 30 May 2011

Keywords:
Elastic rods
Non-linear waves
Homogenization

ABSTRACT

We consider the propagation of a planar loop on a heterogeneous elastic rod with a periodic microstructure consisting of two alternating homogeneous regions with different material properties. The analysis is carried out using a second-order homogenization theory based on a multiple scale asymptotic expansion.

1. Introduction

We consider an inextensible and unshearable elastic rod of circular cross-section and infinite length, that is straight in a stress-free state.

The static and dynamical solutions of such a system under fixed tension and with homogeneous material properties have been extensively studied and classified starting with the classical work of Euler and Kirchhoff, among others [1]. Owing to the remarkable analogy between the dynamical solutions of the spinning tops and the static solutions of this elastic rod, a solution connecting asymptotically the two straight states can be simply expressed in terms of the curvature by a sech function (this solution corresponds to the homoclinic solution connecting the upward pendulum to itself). As planar traveling-wave-reduced equations are formally equivalent to static ones [2], this loop-like solution can also travel along the rod with constant velocity and represent the propagation of localized flexural waves.

A classical problem in the theory of homogenization is to consider longitudinal waves in a heterogeneous elastic medium with a periodic material microstructure of two alternating homogeneous materials. For the case of small-amplitude linear elastic wave, this problem has been analyzed using a homogenization technique [3], based on a multiple scale expansion, introducing a fast length variable on the scale of the microstructure. The leading-order balance yields the effective homogeneous material properties, which is a crude approximation considering that it does not exhibit the dispersive behavior characteristic of the heterogeneous material, brought about by successive reflections on material interfaces. Dispersion is then captured by higher order corrections.

The goal of this paper is to investigate the effect of such heterogeneities in localized flexural waves on a straight elastic rod.

2. Governing equations

Geometrically, the rod is characterized by a curve called the *centerline*, and parametrized by the arc length s. We assume that the rod is inextensible, that is the parametrization $r(s)$ of the centerline is arc-length-preserving for all time, and unshearable, i.e. the cross-sections remain normal to the centerline tangent. Moreover, we assume that the rod is confined to the x–y plane and ignore the possible effect of self-contact. Let (x,y) be the coordinates of a point of the rod centerline, (F,G) the coordinates of the force acting at that point, and Φ the angle the tangent vector at (x,y) makes with the x-axis. The dynamics of the rod is then governed by the following system of equations (cf. e.g. [4]):

$$\rho A x_{tt} = F_s, \tag{1a}$$

$$\rho A y_{tt} = G_s, \tag{1b}$$

$$\rho I \Phi_{tt} = (EI\Phi_s)_s + G\cos\Phi - F\sin\Phi, \tag{1c}$$

$$x_s = \cos\Phi, \tag{1d}$$

$$y_s = \sin\Phi, \tag{1e}$$

where A and I are the cross-section area and second moment of area, ρ is the (mass) density, and E is the Young modulus. We eliminate x and y from the first two equations by differentiating them with respect to s and using the last two equations differentiated twice with respect to t. Thus we obtain the following system for (F,G,Φ):

* Corresponding author.
E-mail address: goriely@maths.ox.ac.uk (A. Goriely).

0020-7462/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2011.05.005

$$A(\cos\Phi)_{tt} = \left(\frac{F_s}{\rho}\right)_s, \tag{2a}$$

$$A(\sin\Phi)_{tt} = \left(\frac{G_s}{\rho}\right)_s, \tag{2b}$$

$$\rho I\Phi_{tt} = (EI\Phi_s)_s + G\cos\Phi - F\sin\Phi. \tag{2c}$$

We assume that the rod is uniform with a circular cross-section of radius R, hence

$$A = \pi R^2, \quad I = \frac{\pi R^4}{4}. \tag{3}$$

The scaling

$$[s] = \frac{R}{2}, \quad [t] = 1s, \quad [F] = [G] = \pi R^2 [E], \quad [E] = \frac{R^2}{4}\frac{[\rho]}{[t]^2} \tag{4}$$

yields the following non-dimensionalized system (all variables are now dimensionless, but are denoted by the same symbol as their dimensional counterparts):

$$(\cos\Phi)_{tt} = \left(\frac{F_s}{\rho}\right)_s, \tag{5a}$$

$$(\sin\Phi)_{tt} = \left(\frac{G_s}{\rho}\right)_s, \tag{5b}$$

$$\rho\Phi_{tt} = (E\Phi_s)_s + G\cos\Phi - F\sin\Phi. \tag{5c}$$

3. Multiple scales asymptotic expansion

We consider a rod with variable material properties on a small scale, so that regions of two different constant properties alternate periodically (see Fig. 1). We denote the length of the unit cell by ε. This cell is composed of two *subdomains* with lengths $\alpha\varepsilon$ and $(1-\alpha)\varepsilon$, densities ρ_a and ρ_b, and the Young moduli E_a and E_b, respectively.

Assuming that the solution to the system (5) is essentially constant over a unit cell, i.e. that ε is a small parameter with respect to a characteristic length of the solution, we introduce a fast length scale $\hat{s}$:

$$\hat{s} = \frac{s}{\varepsilon}, \tag{6}$$

and proceed with a standard multiple scale analysis (see, e.g. [5]). The material parameters are functions of the fast arc length $\hat{s}$ only

$$\rho \equiv \rho(\hat{s}) := \begin{cases} \rho_a, & \hat{s} \in [0,\alpha), \\ \rho_b, & \hat{s} \in [\alpha,1), \end{cases} \tag{7a}$$

$$E \equiv E(\hat{s}) := \begin{cases} E_a, & \hat{s} \in [0,\alpha), \\ E_b, & \hat{s} \in [\alpha,1). \end{cases} \tag{7b}$$

The periodic structure of the rod induces periodicity in terms of the fast arc length variable $\hat{s}$ with period 1 (size of the unit cell in terms of $\hat{s}$) in the dependent variables in the system,

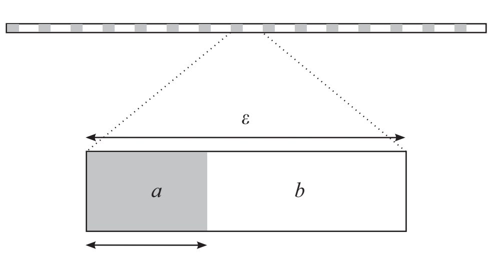

Fig. 1. Heterogeneous rod microstructure.

$$F \equiv F(s,\hat{s},t), \quad G \equiv G(s,\hat{s},t), \quad \Phi \equiv \Phi(s,\hat{s},t). \tag{8}$$

With the addition of the fast arc length variable, the spatial differential operator needs to be modified:

$$\partial_s \mapsto \partial_s + \frac{1}{\varepsilon}\partial_{\hat{s}}, \tag{9}$$

and the system is

$$(\cos\Phi)_{tt} = \left(\frac{F_s + \frac{1}{\varepsilon}F_{\hat{s}}}{\rho}\right)_s + \frac{1}{\varepsilon}\left(\frac{F_s + \frac{1}{\varepsilon}F_{\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{10a}$$

$$(\sin\Phi)_{tt} = \left(\frac{G_s + \frac{1}{\varepsilon}G_{\hat{s}}}{\rho}\right)_s + \frac{1}{\varepsilon}\left(\frac{G_s + \frac{1}{\varepsilon}G_{\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{10b}$$

$$\rho\Phi_{tt} = \left(E\left(\Phi_s + \frac{1}{\varepsilon}\Phi_{\hat{s}}\right)\right)_s + \frac{1}{\varepsilon}\left(E\left(\Phi_s + \frac{1}{\varepsilon}\Phi_{\hat{s}}\right)\right)_{\hat{s}} + G\cos\Phi - F\sin\Phi. \tag{10c}$$

We formally expand the variables in ε,

$$F = \sum_{i=0}^{\infty} \varepsilon^i F_i(s,\hat{s},t), \quad G = \sum_{i=0}^{\infty} \varepsilon^i G_i(s,\hat{s},t), \quad \Phi = \sum_{i=0}^{\infty} \varepsilon^i \Phi_i(s,\hat{s},t), \tag{11}$$

and expand the trigonometric functions on the left-hand sides about Φ_0, e.g.:

$$\cos\Phi = \cos\Phi_0 + \sum_{i=1}^{\infty} \frac{\cos^{(i)}\Phi_0}{i!}\left(\sum_{j=1}^{\infty} \varepsilon^j \Phi_j\right)^i. \tag{12}$$

3.1. $O(\varepsilon^{-2})$ system

Collecting terms in the system (10) expanded via (11) and (12), the lowest order $O(\varepsilon^{-2})$ yields the following system:

$$\left(\frac{F_{0,\hat{s}}}{\rho}\right)_{\hat{s}} = 0, \tag{13a}$$

$$\left(\frac{G_{0,\hat{s}}}{\rho}\right)_{\hat{s}} = 0, \tag{13b}$$

$$(E\Phi_{0,\hat{s}})_{\hat{s}} = 0. \tag{13c}$$

Multiplying (13a) by F_0 and integrating by parts with respect to $\hat{s}$,

$$F_0 \frac{F_{0,\hat{s}}}{\rho}\bigg|_0^1 - \int_0^1 \frac{F_{0,\hat{s}}^2}{\rho} \,\mathrm{d}\hat{s} = 0, \tag{14}$$

the first term vanishes by periodicity of F_0, and, as expected, we conclude that $F_{0,\hat{s}} \equiv 0$, i.e. that F_0 is a function of s and t only. Eqs. (13b) and (13c) yield analogous results for G_0 and Φ_0, thus

$$F_0 \equiv f_0(s,t), \quad G_0 \equiv g_0(s,t), \quad \Phi_0 \equiv \phi_0(s,t). \tag{15}$$

We consistently denote variables that do not explicitly depend on the fast arc length $\hat{s}$ with lowercase letters, and reserve uppercase letters for variables that depend on the rod microstructure.

3.2. $O(\varepsilon^{-1})$ system

The next order of ε in the system (10) expanded via (11) is $O(\varepsilon^{-1})$:

$$\frac{F_{0,s\hat{s}}}{\rho} + \left(\frac{F_{0,s} + F_{1,\hat{s}}}{\rho}\right)_{\hat{s}} = 0, \tag{16a}$$

$$\frac{G_{0,s\hat{s}}}{\rho} + \left(\frac{G_{0,s} + G_{1,\hat{s}}}{\rho}\right)_{\hat{s}} = 0, \tag{16b}$$

$$E\Phi_{0,s\hat{s}} + (E\Phi_{0,s} + E\Phi_{1,\hat{s}})_{\hat{s}} = 0. \tag{16c}$$

We solve Eq. (16a) for F_1 using the following ansatz (cf. [3,6]):

$$F_1(s,\hat{s},t) = f_1(s,t) + K(\hat{s}) f_{0,s}(s,t), \tag{17}$$

where this decomposition is made unique by imposing the following normalization condition:

$$\langle F_1 \rangle = f_1(s,t) \Rightarrow \langle K \rangle = 0, \tag{18}$$

where the operator $\langle \cdot \rangle$ averages over the unit cell:

$$\langle \cdot \rangle : \varphi \mapsto \int_0^1 \varphi(x)\, dx. \tag{19}$$

Eq. (16a) then implies the following ODE for K:

$$\left(\frac{1+K_{\hat{s}}}{\rho} \right)_{\hat{s}} = 0. \tag{20}$$

Recall that ρ is a piecewise-constant function (cf. (7a)). Integrating (20) over each subdomain, we obtain affine functions that we denote K_a and K_b, respectively. The four integration constants (two on each subdomain) are found from the following conditions:

(a) continuity (in terms of $\hat{s}$) of F_1,
(b) periodicity (in terms of $\hat{s}$) of F_1,
(c) the normalization condition (18),
(d) continuity of the parenthesized expression in (20), which we term the *validation condition*.

Conditions (a) and (b) imply continuity and periodicity of K, respectively, yielding $K_a(\alpha) = K_b(\alpha)$ and $K_a(0) = K_b(1)$. Note that the validation condition (d) implies differentiability, since the derivative of the expression vanishes on both intervals, hence on both sides of the point $\hat{s} = \alpha$. The four conditions yield the following solution:

$$K(\hat{s}) = \begin{cases} K_a(\hat{s}) := \dfrac{(1-\alpha)(\rho_a - \rho_b)}{\alpha\rho_a + (1-\alpha)\rho_b}\left(\hat{s} - \dfrac{\alpha}{2}\right), & \hat{s} \in [0,\alpha), \\ K_b(\hat{s}) := \dfrac{\alpha(\rho_a - \rho_b)}{\alpha\rho_a + (1-\alpha)\rho_b}\left(\dfrac{1+\alpha}{2} - \hat{s}\right), & \hat{s} \in [\alpha,1). \end{cases} \tag{21}$$

By symmetry, Eq. (16b) yields

$$G_1(s,\hat{s},t) = g_1(s,t) + K(\hat{s}) g_{0,s}(s,t), \tag{22}$$

where K is also given by (21), and $g_1(s,t) \equiv \langle G_1 \rangle$.

The solution to (16c) is entirely analogous. The ansatz

$$\Phi_1(s,\hat{s},t) = \phi_1(s,t) + L(\hat{s}) \phi_{0,s}(s,t), \tag{23}$$

with the normalization condition

$$\langle \Phi_1 \rangle = \phi_1(s,t) \Rightarrow \langle L \rangle = 0 \tag{24}$$

yields the following ODE for L:

$$(E(1+L_{\hat{s}}))_{\hat{s}} = 0 \tag{25}$$

with the following solution:

$$L(\hat{s}) = \begin{cases} L_a(\hat{s}) := \dfrac{(1-\alpha)(E_b - E_a)}{(1-\alpha)E_a + \alpha E_b}\left(\hat{s} - \dfrac{\alpha}{2}\right), & \hat{s} \in [0,\alpha), \\ L_b(\hat{s}) := \dfrac{\alpha(E_b - E_a)}{(1-\alpha)E_a + \alpha E_b}\left(\dfrac{1+\alpha}{2} - \hat{s}\right), & \hat{s} \in [\alpha,1). \end{cases} \tag{26}$$

For future reference, we note that Eqs. (20) and (25) along with the validation conditions imply that the differentiated expressions are constant over the unit cell. We can evaluate these constants using the solutions (21) and (26) for K and L, respectively:

$$\frac{1+K_{\hat{s}}}{\rho} \equiv \frac{1}{\alpha\rho_a + (1-\alpha)\rho_b} = \langle \rho \rangle^{-1} =: \rho_h^{-1}, \tag{27a}$$

$$E(1+L_{\hat{s}}) \equiv \frac{E_a E_b}{(1-\alpha)E_a + \alpha E_b} = \langle E^{-1} \rangle^{-1} =: E_h. \tag{27b}$$

3.3. $O(\varepsilon^0)$ system

Next, we consider the system of order $O(\varepsilon^0)$:

$$(\cos\Phi_0)_{tt} = \left(\frac{F_{0,s} + F_{1,\hat{s}}}{\rho} \right)_s + \left(\frac{F_{1,s} + F_{2,\hat{s}}}{\rho} \right)_{\hat{s}}, \tag{28a}$$

$$(\sin\Phi_0)_{tt} = \left(\frac{G_{0,s} + G_{1,\hat{s}}}{\rho} \right)_s + \left(\frac{G_{1,s} + G_{2,\hat{s}}}{\rho} \right)_{\hat{s}}, \tag{28b}$$

$$\rho\Phi_{0,tt} = (E(\Phi_{0,s} + \Phi_{1,\hat{s}}))_s + (E(\Phi_{1,s} + \Phi_{2,\hat{s}}))_{\hat{s}} + G_0\cos\Phi_0 - F_0\sin\Phi_0. \tag{28c}$$

Using the ansatz (17), (22), (23) for F_1, G_1, Φ_1, as well as identities (27), the system (28) becomes

$$(\cos\phi_0)_{tt} = \frac{f_{0,ss}}{\rho_h} + \left(\frac{f_{1,s} + K f_{0,ss} + F_{2,\hat{s}}}{\rho} \right)_{\hat{s}}, \tag{29a}$$

$$(\sin\phi_0)_{tt} = \frac{g_{0,ss}}{\rho_h} + \left(\frac{g_{1,s} + K g_{0,ss} + G_{2,\hat{s}}}{\rho} \right)_{\hat{s}}, \tag{29b}$$

$$\rho\phi_{0,tt} = E_h \phi_{0,ss} + (E(\phi_{1,s} + L\phi_{0,ss} + \Phi_{2,\hat{s}}))_{\hat{s}} + g_0\cos\phi_0 - f_0\sin\phi_0. \tag{29c}$$

3.3.1. Averaged $O(\varepsilon^0)$ system

We apply the averaging operator $\langle \cdot \rangle$ (19) on the system (29). We note that $\langle \varphi_{\hat{s}} \rangle \equiv 0$ for any function φ periodic on a unit cell, hence the second terms on the right-hand sides of (29) all vanish when averaged. The $O(\varepsilon^0)$ balance is thus

$$(\cos\phi_0)_{tt} = \frac{f_{0,ss}}{\rho_h}, \tag{30a}$$

$$(\sin\phi_0)_{tt} = \frac{g_{0,ss}}{\rho_h}, \tag{30b}$$

$$\rho_h\phi_{0,tt} = E_h\phi_{0,ss} + g_0\cos\phi_0 - f_0\sin\phi_0. \tag{30c}$$

This is a system describing the homogenized behavior of the heterogeneous rod in the leading-order approximation. It has the form of a system of equations describing a homogeneous rod (cf. (5)), where the constant material properties are the bulk density ρ_h, and E_h, which is one-half of the harmonic average of the Young modulus (cf. (27)). Up to now, the analysis of the system was general. We now focus on the localized flexural waves in order to understand the effect on the microstructure in their characteristics. To do so, we solve the system (30) by a traveling wave reduction:

$$\xi = s - ct, \quad \partial_t \mapsto -c\partial_\xi, \quad \partial_s \mapsto \partial_\xi. \tag{31}$$

The reduced system is

$$c^2\rho_h(\cos\phi_0)_{\xi\xi} = f_{0,\xi\xi}, \tag{32a}$$

$$c^2\rho_h(\sin\phi_0)_{\xi\xi} = g_{0,\xi\xi}, \tag{32b}$$

$$(c^2\rho_h - E_h)\phi_{0,\xi\xi} = g_0\cos\phi_0 - f_0\sin\phi_0. \tag{32c}$$

The solutions we are looking for are loops on a straight rod with a tension T applied at infinity. We thus impose the following boundary conditions at infinity for the force:

$$\lim_{\xi \to \pm\infty} f_0 = T, \quad \lim_{\xi \to \pm\infty} g_0 = 0, \tag{33}$$

while the boundary conditions for the angle for a single loop are

$$\lim_{\xi\to-\infty}\phi_0=0,\quad \lim_{\xi\to+\infty}\phi_0=2\pi. \tag{34}$$

Integrating twice Eqs. (32a) and (32b) subject to the above boundary conditions, we have

$$f_0=c^2\rho_h(\cos\phi_0-1)+T, \tag{35a}$$

$$g_0=c^2\rho_h\sin\phi_0. \tag{35b}$$

Now we can eliminate f_0 and g_0 from (32c):

$$\phi_{0,\xi\xi}=\frac{1}{\ell^2}\sin\phi_0, \tag{36}$$

where

$$\ell^2 := \frac{E_h-c^2\rho_h}{T-c^2\rho_h}. \tag{37}$$

As expected from the Kirchhoff analogy, Eq. (36) is the pendulum equation where the tangent angle plays the role of the angle the pendulum makes with the vertical and arc length corresponds to time [1]. The boundary conditions (34) correspond to a homoclinic orbit, with $\phi_0=0 \bmod 2\pi$ as the homoclinic point. We therefore conclude that $\ell^2>0$ (a negative value of ℓ^2 would have $\phi_0=\pi$ for a homoclinic point), which implies the following condition on the wave speed:

$$c^2\in\mathbb{R}^+\backslash[c_1^2,c_2^2], \tag{38a}$$

$$c_1^2 := \min\left(\frac{E_h}{\rho_h},\frac{T}{\rho_h}\right),\quad c_2^2 := \max\left(\frac{E_h}{\rho_h},\frac{T}{\rho_h}\right). \tag{38b}$$

Note that $\ell\to0$ when the wave speed approaches $c_0 := \sqrt{E_h/\rho_h}$ (speed of sound in a homogeneous material with the Young modulus E_h and density ρ_h), and $\ell\to\infty$ for $c^2\to T/\rho_h$. With zero tension, the parameter $\ell\equiv\sqrt{1-c_0^2/c^2}$ is an increasing function of the wave speed c, and (38) yields a lower bound c_0 for the wave speed. Therefore, we have $0<\ell<1$, where $\ell\to0$ for $c\to c_0$, and $\ell\to1$ for $c\to\infty$. The solution to (36) with boundary conditions (34) is the well-known homoclinic orbit of the pendulum (cf. [1]),

$$\phi_0(\xi)=4\arctan(e^{(\xi-\xi_0)/\ell}), \tag{39}$$

where ξ_0 is an integration constant that corresponds to the position of the midpoint of the loop. The parameter ℓ can now be identified as the characteristic size of the loop. The solution (39) is shown in Fig. 2. The shape of the planar rod corresponding to this tangent angle is a single loop that straightens out exponentially on the two ends, and is shown in Fig. 5.

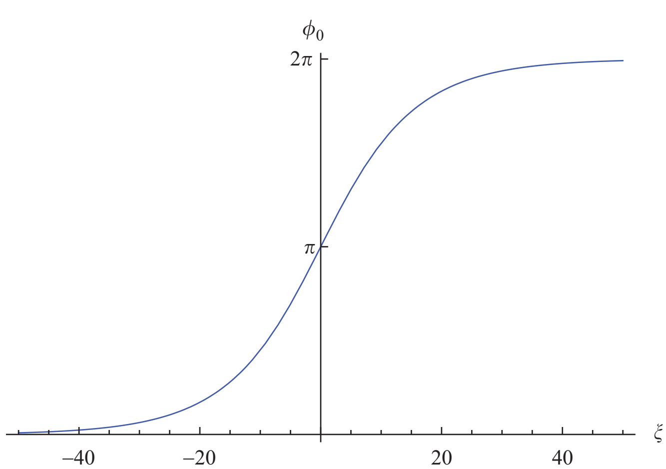

Fig. 2. Homogeneous rod solution (39) for the angle ($\xi_0=0$, $\ell=10$).

3.3.2. Ansatz for F_2, G_2, Φ_2

We now go back to the $O(\varepsilon^0)$ system (29). The averaged balance (30) implies

$$\left\langle\frac{f_{1,s}+Kf_{0,ss}+F_{2,\hat{s}}}{\rho}\right\rangle_{\hat{s}}=0, \tag{40a}$$

$$\left\langle\frac{g_{1,s}+Kg_{0,ss}+G_{2,\hat{s}}}{\rho}\right\rangle_{\hat{s}}=0, \tag{40b}$$

$$(E(\phi_{1,s}+L\phi_{0,ss}+\Phi_{2,\hat{s}}))_{\hat{s}}=(\rho-\rho_h)\phi_{0,tt}. \tag{40c}$$

Following the ansatz in (17), we decompose F_2, G_2, and Φ_2 as follows:

$$F_2(s,\hat{s},t)=f_2(s,t)+K(\hat{s})f_{1,s}(s,t)+M(\hat{s})f_{0,ss}(s,t), \tag{41a}$$

$$G_2(s,\hat{s},t)=g_2(s,t)+K(\hat{s})g_{1,s}(s,t)+M(\hat{s})g_{0,ss}(s,t), \tag{41b}$$

$$\Phi_2(s,\hat{s},t)=\phi_2(s,t)+L(\hat{s})\phi_{1,s}(s,t)+N(\hat{s})\phi_{0,ss}(s,t), \tag{41c}$$

subject to the following normalization:

$$\langle M\rangle=0, \tag{42a}$$

$$\langle N\rangle=0, \tag{42b}$$

which is equivalent to

$$\langle F_2\rangle=f_2(s,t),\quad \langle G_2\rangle=g_2(s,t),\quad \langle\Phi_2\rangle=\phi_2(s,t). \tag{43}$$

Eq. (40a) becomes

$$\left\langle\frac{1+K_{\hat{s}}}{\rho}\right\rangle_{\hat{s}}f_{1,s}(s,t)+\left\langle\frac{K+M_{\hat{s}}}{\rho}\right\rangle_{\hat{s}}f_{0,ss}(s,t)=0. \tag{44}$$

The first term vanishes by (20), leaving

$$\left\langle\frac{K+M_{\hat{s}}}{\rho}\right\rangle_{\hat{s}}=0. \tag{45}$$

We solve this equation for M analogously to (20), by integrating over the two subdomains separately. Recall that on each subdomain ρ is constant (cf. (7a)), while K is an affine function (cf. (21)). We thus obtain two quadratic functions M_a, M_b, where the integration constants are obtained from

(a) continuity (in terms of $\hat{s}$) of F_2,
(b) periodicity (in terms of $\hat{s}$) of F_2,
(c) the normalization condition (42a)
(d) the validation condition: continuity of the parenthesized expression in (45).

The solution for M is

$$M(\hat{s})=\begin{cases} M_a(\hat{s}) := -\dfrac{(1-\alpha)(\rho_a-\rho_b)}{2\rho_h}\left(\hat{s}^2-\alpha\hat{s}+\dfrac{\alpha(2\alpha-1)}{6}\right), & \hat{s}\in[0,\alpha),\\ M_b(\hat{s}) := \dfrac{\alpha(\rho_a-\rho_b)}{2\rho_h}\left(\hat{s}^2-(1+\alpha)\hat{s}+\dfrac{2\alpha^2+3\alpha+1}{6}\right), & \hat{s}\in[\alpha,1).\end{cases} \tag{46}$$

Similarly, we find N from (40c),

$$(\rho-\rho_h)\phi_{0,tt}=(E(L+N_{\hat{s}}))_{\hat{s}}\phi_{0,ss}. \tag{47}$$

Noting that ϕ_0 satisfies the wave equation with speed c, this leads to

$$(E(L+N_{\hat{s}}))_{\hat{s}}=c^2(\rho-\rho_h). \tag{48}$$

Integrating the equation on the two subdomains, where E and ρ are constants, as L is affine function on each, we obtain two quadratic functions N_a and N_b. The constants of integration are obtained by imposing the same four conditions as above. We note, however, that the validation condition, i.e. the continuity of $(E(L+N_{\hat{s}}))$ here does not implies differentiability, as the right-hand side of (48) has a jump at the material interface $\hat{s}=\alpha$. Therefore, the solution obtained for N is a weak solution.

$$N(\hat{s}) = \begin{cases} N_a(\hat{s}) := n_{a2}\hat{s}^2 + n_{a1}\hat{s} + n_{a0}, & \hat{s} \in [0,\alpha), \\ N_b(\hat{s}) := n_{b2}\hat{s}^2 + n_{b1}\hat{s} + n_{b0}, & \hat{s} \in [\alpha,1), \end{cases} \tag{49}$$

where the n's are constants that depend on the parameters $\alpha,\rho_a,\rho_b,E_a,E_b$, and the wave speed c. (Henceforth, we omit the explicit expressions for the coefficients as they become rather cumbersome at this point.)

3.4. $O(\varepsilon)$ system

The next order system we consider is $O(\varepsilon)$:

$$-(\Phi_1 \sin\Phi_0)_{tt} = \left(\frac{F_{1,s}+F_{2,\hat{s}}}{\rho}\right)_s + \left(\frac{F_{2,s}+F_{3,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{50a}$$

$$(\Phi_1 \cos\Phi_0)_{tt} = \left(\frac{G_{1,s}+G_{2,\hat{s}}}{\rho}\right)_s + \left(\frac{G_{2,s}+G_{3,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{50b}$$

$$\begin{aligned}\rho\Phi_{1,tt} = {} & \big(E(\Phi_{1,s}+\Phi_{2,\hat{s}})\big)_s + \big(E(\Phi_{2,s}+\Phi_{3,\hat{s}})\big)_{\hat{s}} + G_1\cos\Phi_0 \\ & -G_0\Phi_1\sin\Phi_0 - F_1\sin\Phi_0 - F_0\Phi_1\cos\Phi_0.\end{aligned} \tag{50c}$$

Applying the ansatz for (F_1,G_1,Φ_1) and (F_2,G_2,Φ_2), as well as the identity (27a) and the following one obtained from (45) and the solutions for K (21) and M (46):

$$\frac{K+M_{\hat{s}}}{\rho} \equiv 0, \tag{51}$$

the system (50) becomes

$$-((\phi_1 + L\phi_{2,s})\sin\phi_0)_{tt} = \frac{f_{1,ss}}{\rho_h} + \left(\frac{f_{2,s}+Kf_{1,ss}+Mf_{0,sss}+F_{3,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{52a}$$

$$((\phi_1 + L\phi_{2,s})\cos\phi_0)_{tt} = \frac{g_{1,ss}}{\rho_h} + \left(\frac{g_{2,s}+Kg_{1,ss}+Mg_{0,sss}+G_{3,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{52b}$$

$$\begin{aligned}\rho(\phi_{1,tt}+L\phi_{0,stt}) = {} & E_h\phi_{1,ss} + E(L+N_{\hat{s}})\Phi_{0,sss} \\ & + \big(E(\phi_{2,s}+L\phi_{1,ss}+N\phi_{0,sss}+\Phi_{3,\hat{s}})\big)_{\hat{s}} \\ & + (g_1+Kg_{0,s})\cos\phi_0 - g_0(\phi_1+L\phi_{0,s})\sin\phi_0 \\ & -(f_1+Kf_{0,s})\sin\phi_0 - f_0(\phi_1+L\phi_{0,s})\cos\phi_0.\end{aligned} \tag{52c}$$

3.4.1. Averaged $O(\varepsilon)$ system

Averaging the system (50) over the unit cell, and using the following identities:

$$\langle E(L+N_{\hat{s}})\rangle = 0, \tag{53a}$$

$$\langle \rho L\rangle = 0, \tag{53b}$$

we obtain the $O(\varepsilon)$ balance:

$$-\rho_h(\phi_1\sin\phi_0)_{tt} = f_{1,ss}, \tag{54a}$$

$$\rho_h(\phi_1\cos\phi_0)_{tt} = g_{1,ss}, \tag{54b}$$

$$\rho_h\phi_{1,tt} = E_h\phi_{1,ss} - \phi_1(g_0\sin\phi_0 + f_0\cos\phi_0) + g_1\cos\phi_0 - f_1\sin\phi_0. \tag{54c}$$

As the zeroth-order solution (f_0,g_0,ϕ_0) is known (cf. (35a), (35b), (39)), this is a system of equations for (f_1,g_1,ϕ_1). Applying a traveling wave reduction, and integrating the first two equations with zero boundary conditions at infinity, we have

$$f_1 = -c^2\rho_h\phi_1\sin\phi_0, \tag{55a}$$

$$g_1 = c^2\rho_h\phi_1\cos\phi_0. \tag{55b}$$

Eliminating f_1,g_1 from the reduced third equation yields

$$\phi_{1,\xi\xi} = \frac{1}{\ell^2}\phi_1\cos\phi_0, \tag{56}$$

where ℓ is given by (37). The solution of (56) satisfying the boundary conditions is given by

$$\phi_1(\xi) = \phi_{0,\xi}(\xi) = \frac{2}{\ell}\operatorname{sech}\frac{\xi-\xi_0}{\ell}, \tag{57}$$

where ϕ_0 is the zeroth-order solution (39). This is the first correction in the homogenized solution, one that captures (the leading order of) the difference with respect to the homogeneous system behavior.

3.4.2. Ansatz for F_3,G_3,Φ_3

We now go back to the $O(\varepsilon)$ system (50), with the following ansatz for F_3,G_3,Φ_3:

$$F_3(s,\hat{s},t) = f_3(s,t) + K(\hat{s})f_{2,s}(s,t) + M(\hat{s})f_{1,ss}(s,t) + P(\hat{s})f_{0,sss}(s,t), \tag{58a}$$

$$G_3(s,\hat{s},t) = g_3(s,t) + K(\hat{s})g_{2,s}(s,t) + M(\hat{s})g_{1,ss}(s,t) + P(\hat{s})g_{0,sss}(s,t), \tag{58b}$$

$$\Phi_3(s,\hat{s},t) = \phi_3(s,t) + L(\hat{s})\phi_{2,s}(s,t) + N(\hat{s})\phi_{1,ss}(s,t) + Q(\hat{s})\phi_{0,sss}(s,t), \tag{58c}$$

where P and Q satisfy the normalization conditions

$$\langle P\rangle = 0, \quad \langle Q\rangle = 0, \tag{59a}$$

i.e.

$$f_3(s,t) = \langle F_3\rangle, \quad g_3(s,t) = \langle G_3\rangle, \quad \phi_3(s,t) = \langle \Phi_3\rangle. \tag{60}$$

Using the ansatz for (F_1,G_1,Φ_1) and (F_2,G_2,Φ_2), as well as the solution (57) and the identities (27), and (53), the system (50) becomes

$$L(\cos\phi_0)_{stt} = \left(\frac{M+P_{\hat{s}}}{\rho}\right)_{\hat{s}} f_{0,sss}, \tag{61a}$$

$$L(\sin\phi_0)_{stt} = \left(\frac{M+P_{\hat{s}}}{\rho}\right)_{\hat{s}} g_{0,sss}, \tag{61b}$$

$$\rho L\phi_{0,stt} = [E(L+N_{\hat{s}}) + (E(N+Q_{\hat{s}}))_{\hat{s}}]\phi_{0,sss} + c^2\langle\rho_h\rangle(K - L(1-\cos\phi_0))\phi_{0,s}. \tag{61c}$$

Transforming the left-hand side with the help of the averaged $O(1)$ system (30), each of the first two equations reduces to

$$\langle\rho_h\rangle^{-1}L = \left(\frac{M+P_{\hat{s}}}{\rho}\right)_{\hat{s}}. \tag{62}$$

As L is an affine function, and M is a quadratic one, the solution for P is a cubic function each subdomain,

$$P(\hat{s}) = \begin{cases} P_a(\hat{s}) := p_{a3}\hat{s}^3 + p_{a2}\hat{s}^2 + p_{a1}\hat{s} + p_{a0}, & \hat{s} \in [0,\alpha), \\ P_b(\hat{s}) := p_{b3}\hat{s}^3 + p_{b2}\hat{s}^2 + p_{b1}\hat{s} + p_{b0}, & \hat{s} \in [\alpha,1), \end{cases} \tag{63}$$

where the coefficients p are determined by the usual conditions: continuity, periodicity, normalization and validation, and are functions of the material parameters $E_a,E_b,\rho_a,\rho_b,\alpha$, and the wave speed c.

Since ϕ_0 satisfies the wave equation with wave speed c, Eq. (61c) yields

$$[c^2\rho L - E(L+N_{\hat{s}}) - (E(N+Q_{\hat{s}}))_{\hat{s}}]\phi_{0,sss} = c^2\rho_h(K\cos 2\phi_0 - L(1-\cos\phi_0))\phi_{0,s}. \tag{64}$$

This equation does not yield an ODE for Q because of the extra term on the right-hand side. We therefore amend the ansatz (58c) for Φ_3 so as to cancel this term out. The modified ansatz is

$$\Phi_3(s,\hat{s},t) = \phi_3 + L\phi_{2,s} + N\phi_{1,ss} + Q\phi_{0,sss} + H(s,\hat{s},t), \tag{65}$$

H being a function such that

$$(EH_{\hat{s}})_{\hat{s}} = K\chi(s,t)+L\lambda(s,t), \tag{66}$$

where

$$\chi(s,t) := c^2\rho_h\phi_{0,s}\cos 2\phi_0, \tag{67a}$$

$$\lambda(s,t) := -c^2\rho_h\phi_{0,s}(1-\cos\phi_0). \tag{67b}$$

Moreover, we impose the following normalization condition on H:

$$\langle H\rangle = 0, \tag{68}$$

so that $\phi_3 = \langle \Phi_3\rangle$ still holds, and we require H to be continuous and periodic. Analogously to the validation conditions seen above, we require $EH_{\hat{s}}$ to be continuous as well. We obtain H by integrating equation (66) twice over each of the two subdomains, and determine the integration constants from the aforementioned conditions on H. It is clear that the $\hat{s}$-dependence of H is cubic. Note that integrating equation (66) with respect to $\hat{s}$ leaves χ and λ intact, therefore the form of the solution for H is

$$H(s,\hat{s},t) = H_K(\hat{s})\chi(s,t)+H_L(\hat{s})\lambda(s,t). \tag{69}$$

The new ansatz (65) cancels out the last two terms in the right-hand side of (64), yielding the following ODE for Q:

$$(E(N+Q_{\hat{s}}))_{\hat{s}} = (c^2\rho-E)L-EN_{\hat{s}}. \tag{70}$$

As previously, Q is obtained by integrating twice over each of the two subdomains, and the integration constants are found the usual way. As L is an affine function, and N is quadratic, the resulting function Q is a cubic on each subdomain:

$$Q(\hat{s}) = \begin{cases} Q_a(\hat{s}) := q_{a3}\hat{s}^3+q_{a2}\hat{s}^2+q_{a1}\hat{s}+q_{a0}, & \hat{s}\in[0,\alpha), \\ Q_b(\hat{s}) := q_{b3}\hat{s}^3+q_{b2}\hat{s}^2+q_{b1}\hat{s}+q_{b0}, & \hat{s}\in[\alpha,1). \end{cases} \tag{71}$$

3.5. $O(\varepsilon^2)$ system

Collected terms of order ε^2 yield the following system:

$$\left(-\Phi_2\sin\Phi_0-\frac{1}{2}\Phi_1^2\cos\Phi_0\right)_{tt} = \left(\frac{F_{2,s}+F_{3,\hat{s}}}{\rho}\right)_s+\left(\frac{F_{3,s}+F_{4,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{72a}$$

$$\left(\Phi_2\cos\Phi_0-\frac{1}{2}\Phi_1^2\sin\Phi_0\right)_{tt} = \left(\frac{G_{2,s}+G_{3,\hat{s}}}{\rho}\right)_s+\left(\frac{G_{3,s}+G_{4,\hat{s}}}{\rho}\right)_{\hat{s}}, \tag{72b}$$

$$\begin{aligned}\rho\Phi_{2,tt} = {}& E(\Phi_{2,s}+\Phi_{3,\hat{s}})_s+(E(\Phi_{3,s}+\Phi_{4,\hat{s}}))_{\hat{s}} \\ &+G_0\left(-\Phi_2\sin\Phi_0-\frac{1}{2}\Phi_1^2\cos\Phi_0\right)-G_1\Phi_1\sin\Phi_0+G_2\cos\Phi_0 \\ &-F_0\left(\Phi_2\cos\Phi_0-\frac{1}{2}\Phi_1^2\sin\Phi_0\right)-F_1\Phi_1\cos\Phi_0-F_2\sin\Phi_0.\end{aligned} \tag{72c}$$

3.5.1. Averaged $O(\varepsilon^2)$ system

Using the ansatz expressions and the known identities, the averaged system (72) becomes

$$-(\phi_2\sin\phi_0)_{tt} = \rho_h^{-1}f_{2,ss}+\left\langle\frac{M+P_{\hat{s}}}{\rho}\right\rangle f_{0,ssss}+\frac{1}{2}\langle L^2\rangle(\phi_{0,s}^2\cos\phi_0)_{tt}, \tag{73a}$$

$$(\phi_2\cos\phi_0)_{tt} = \rho_h^{-1}g_{2,ss}+\left\langle\frac{M+P_{\hat{s}}}{\rho}\right\rangle g_{0,ssss}+\frac{1}{2}\langle L^2\rangle(\phi_{0,s}^2\sin\phi_0)_{tt}, \tag{73b}$$

$$\begin{aligned}\rho_h\phi_{2,tt} = {}& E_h\phi_{2,ss}-c^2\langle\rho_h\rangle\phi_2(1-\cos\phi_0)-f_2\sin\phi_0+g_2\cos\phi_0 \\ &-\frac{c^2\rho_h\langle L^2\rangle}{2}\phi_{0,s}^2\sin\phi_0+\langle E(N+Q_{\hat{s}})\rangle\phi_{0,ssss}-\langle\rho N\rangle\phi_{0,sstt} \\ &+\langle EH_{K,\hat{s}}\rangle\chi_s+\langle EH_{L,\hat{s}}\rangle\lambda_s.\end{aligned} \tag{73c}$$

This is a system for (f_2,g_2,ϕ_2), which we solve using a traveling wave reduction. Integrating twice the reduced first two equations with zero boundary conditions at infinity yields

$$f_2 = -c^2\rho_h\phi_2\sin\phi_0-\rho_h\left\langle\frac{M+P_{\hat{s}}}{\rho}\right\rangle f_{0,\xi\xi}-\frac{c^2}{2}\rho_h\langle L^2\rangle\phi_{0,s}^2\cos\phi_0, \tag{74a}$$

$$g_2 = c^2\rho_h\phi_2\cos\phi_0-\rho_h\left\langle\frac{M+P_{\hat{s}}}{\rho}\right\rangle g_{0,\xi\xi}-\frac{c^2}{2}\rho_h\langle L^2\rangle\phi_{0,s}^2\sin\phi_0. \tag{74b}$$

Eliminating f_2 and g_2 from the reduced Eq. (73c) yields the following ODE for $\phi_2(\xi)$:

$$\phi_{2,\xi\xi}-\frac{1}{\ell^2}\phi_2\cos\phi_0 = \psi(\xi), \tag{75}$$

where ℓ is given by (37), and ψ is

$$\begin{aligned}\psi(\xi) = {}& \frac{1}{\ell^2}\left(\frac{\langle E(N+Q_{\hat{s}})\rangle}{c^2\rho_h}-\frac{\langle\rho N\rangle}{\rho_h}\right)\phi_{0,\xi\xi\xi\xi} \\ &+\frac{1}{\ell^2}\left(\langle EH_{K,\hat{s}}\rangle-\langle EH_{L,\hat{s}}\rangle(1-\cos\phi_0)-\rho_h\left\langle\frac{M+P_{\hat{s}}}{\rho}\right\rangle\right)\phi_{0,\xi\xi} \\ &-\frac{1}{\ell^2}\left(\langle EH_{L,\hat{s}}\rangle+\frac{\langle L^2\rangle}{2}\right)\phi_{0,\xi}^2\sin\phi_0.\end{aligned} \tag{76}$$

The homogeneous part of Eq. (75) is the same as the equation for ϕ_1 (56). One solution (satisfying the boundary conditions for ϕ_1) was found to be (57):

$$\phi_{2,hom}^{(1)} = \phi_{0,\xi}. \tag{77}$$

The other solution, found by a reduction $\phi_2(\xi) = u(\xi)\phi_{0,\xi}(\xi)$ of the homogeneous equation, is:

$$\phi_{2,hom}^{(2)} = \phi_{0,\xi}\int\frac{1}{\phi_{0,\xi}^2(\xi)}\,d\xi. \tag{78}$$

A particular solution of Eq. (75) is now obtained by variation of parameters

$$\phi_{2,part} = \phi_{2,hom}^{(1)}\int\frac{W_1(x)}{W(x)}\,dx+\phi_{2,hom}^{(2)}\int\frac{W_2(x)}{W(x)}\,dx, \tag{79}$$

where W is the Wronskian determinant for the homogeneous basis (77), (78), and

$$W_1(\xi) := -\psi(\xi)\phi_{2,hom}^{(2)},\quad W_2(\xi) := \psi(\xi)\phi_{2,hom}^{(1)}. \tag{80}$$

The explicit form of the particular solution ϕ_2 is far too complex to be reproduced here, but it is found to satisfy null boundary conditions at infinity.

4. Homogenized traveling wave solution

Using a homogenization approach, we have obtained a leading-order solution ϕ_0 and the first correction ϕ_1 in terms of the angle variable,

$$\phi_0(\xi) = 4\arctan(e^{(\xi-\xi_0)/\ell}), \tag{39}$$

$$\phi_1(\xi) = \phi_{0,\xi}(\xi) = \frac{2}{\ell}\,\mathrm{sech}\,\frac{\xi-\xi_0}{\ell}, \tag{57}$$

as well as the second correction ϕ_2, given by (79). Whereas ϕ_0 and ϕ_1 depend on the material parameters, the tension T, and the wave speed c only through the characteristic length ℓ (37), the second-order correction explicitly depends on all material parameters E_a, E_b, ρ_a, ρ_b, α, and the wave speed c (note the

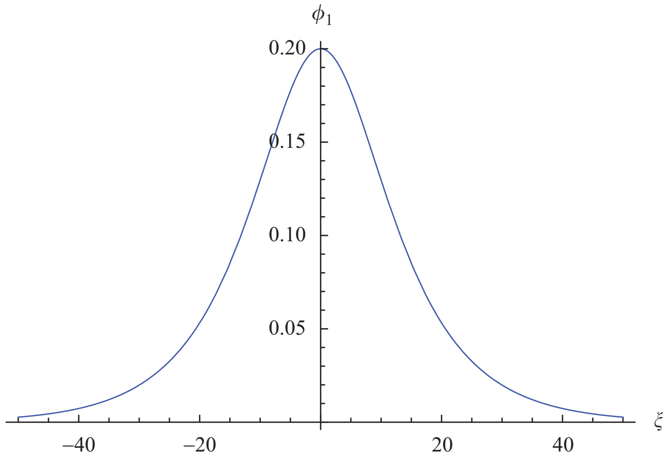

Fig. 3. First-order correction (57) in terms of the angle ($\xi_0 = 0$, $\ell = 10$).

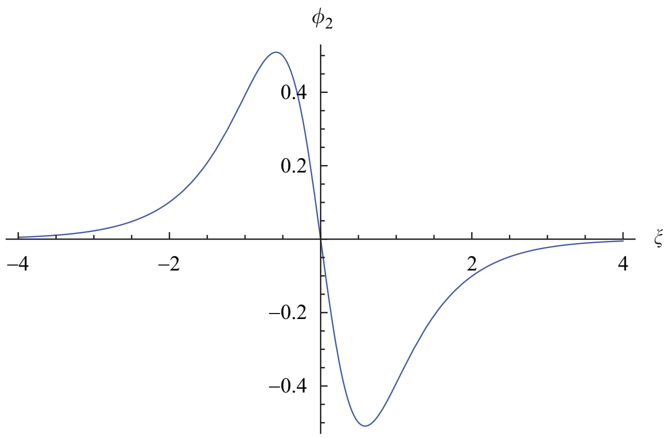

Fig. 4. Numeric solution for the second-order correction (79) in terms of the angle ($\xi_0 = 0$, $\ell = 10$, $E_1 = 1$, $E_2 = \frac{1}{2}$, $\rho_1 = 0.8$, $\rho_2 = 1$, $\alpha = 0.2$, $T = 1$, and the wave speed c is given by (37)).

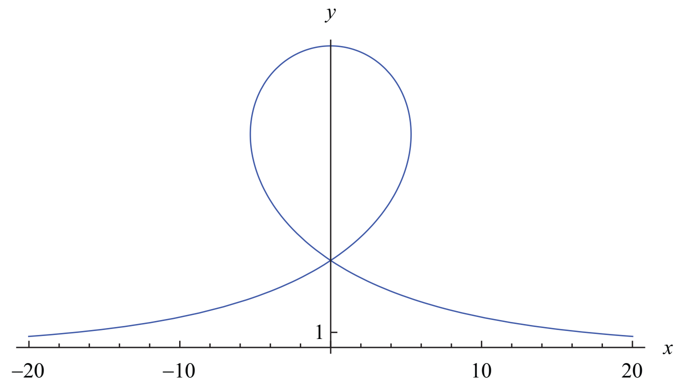

Fig. 5. Loop-like traveling wave solution for the homogenous rod, corresponding to the solution (39) for the angle ($\xi_0 = 0$, $\ell = 10$).

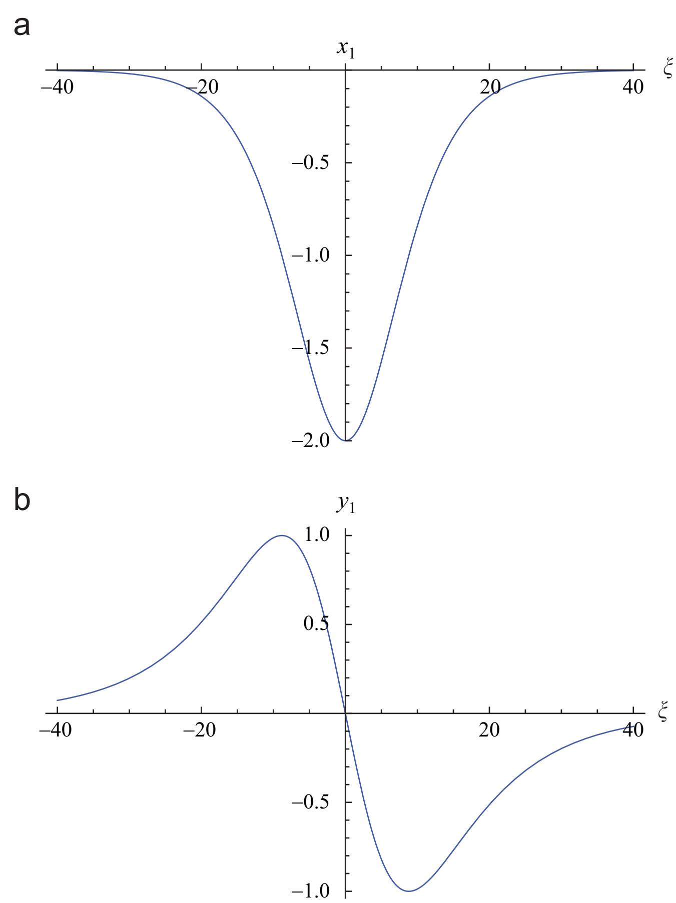

Fig. 6. First correction traveling wave solution (83a) ($\xi_0 = 0, \ell = 10$).

dependence of ψ on various averaged quantities in (76), each being an expression involving the material parameters).

The graphs of the three solutions are shown in Figs. 2–4. The effect of the first correction is to increase the angle ϕ within a localized region, which coincides with the extent of the loop (compare with Fig. 2).

In order to view the solution in terms of the shape of the rod in the (x,y) plane, rather than integrating the x and y equations (1d) and (1e) for the combined angle $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$, we carry out the same multiple scale expansion as seen above for these two equations.

4.1. Leading-order solution

Collected terms of order $O(\varepsilon^0)$ for the Cartesian coordinates system and averaging over the unit cell yields

$$x_{0,s} = \cos\phi_0, \tag{81a}$$

$$y_{0,s} = \sin\phi_0. \tag{81b}$$

The solution of the zero-order system (82a) is the well-known loop solution on a homogeneous rod, depicted in Fig. 5.

4.2. First-order correction

Collecting terms of order $O(\varepsilon)$ and applying the averaging operator yields

$$x_{1,s} = -\phi_1 \sin\phi_0, \tag{82a}$$

$$y_{1,s} = \phi_1 \cos\phi_0, \tag{82b}$$

This gives the following solution in terms of the Cartesian coordinates:

$$x_1(\xi) = \cos(4\arctan(e^{(\xi-\xi_0)/\ell})) - 1, \tag{83a}$$

$$y_1(\xi) = \sin(4\arctan(e^{(\xi-\xi_0)/\ell})), \tag{83b}$$

where the integration constants have been set by null Dirichlet boundary conditions at infinity. The graphs of the two coordinate solutions are shown in Fig. 6. The effect of the first correction in the x direction is to move the homogeneous solution loop to the left (see Fig. 6(a)). In the y direction, the "front part" of the loop, i.e. the part corresponding to values of the independent variable $\xi < \xi_0$, experiences a shift upwards, while the $\xi > \xi_0$ part shifts downwards (see Fig. 6). This effect is illustrated in Fig. 7, where relatively large values of ε have been used in order to accentuate the effect. The first-order correction conserves the arc length of the loop, since $\lim_{\xi\to\infty} x_1(\xi) = \lim_{\xi\to-\infty} x_1(\xi)$.

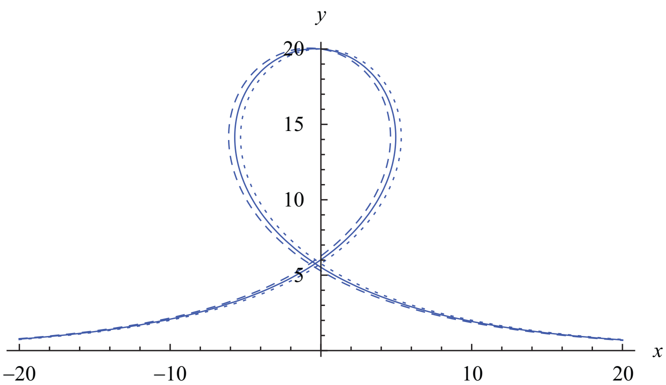

Fig. 7. Homogenized solution up to the first correction, $\phi = \phi_0 + \varepsilon\phi_1$, shown in the Cartesian plane for two different values of ε: $\varepsilon = \frac{3}{8}$ (solid curve) and $\varepsilon = \frac{3}{4}$ (dashed curve). The homogeneous solution ($\varepsilon = 0$) is shown in dotted ($\xi_0 = 0$, $\ell = 10$).

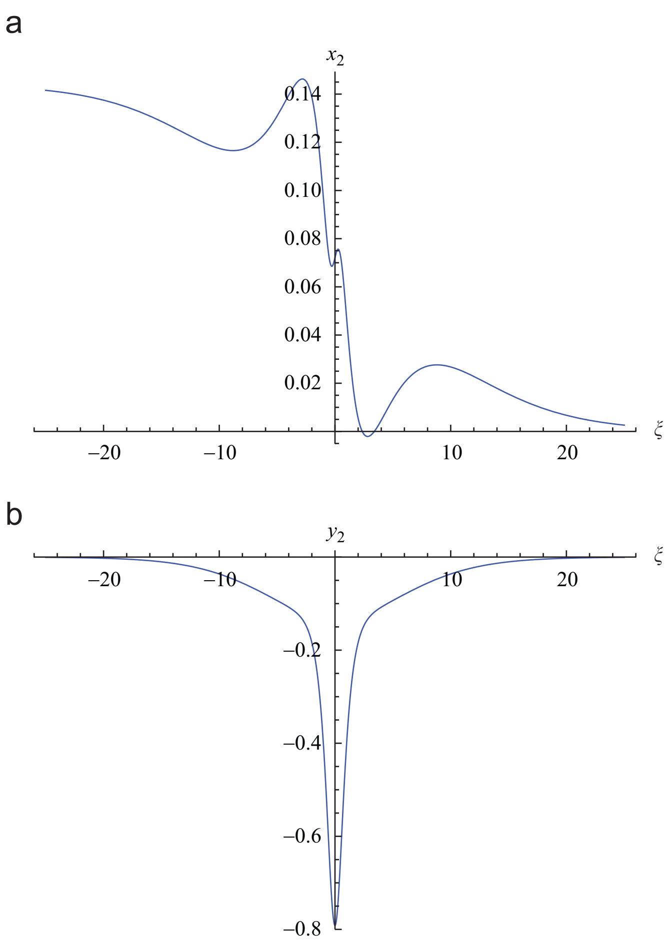

Fig. 8. Second correction to the traveling wave solution for the Cartesian coordinates, (84) ($\xi_0 = 0$, $\ell = 10$, $E_1 = 1$, $E_2 = \frac{1}{2}$, $\rho_1 = 0.8$, $\rho_2 = 1$, $\alpha = 0.2$, $T = 1$, and the wave speed c is given by (37)).

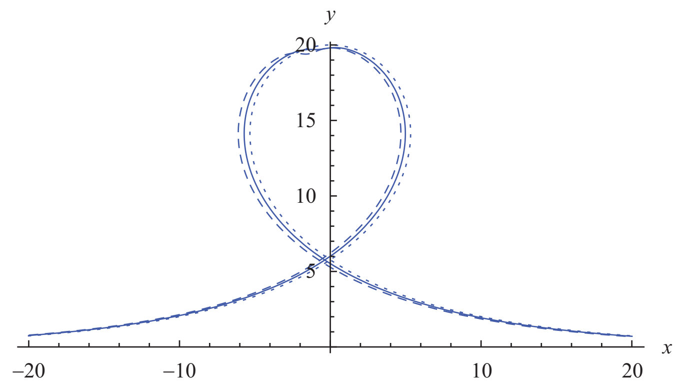

Fig. 9. Homogenized solution up to the second correction, $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$, shown in the Cartesian plane for two different values of ε: $\varepsilon = \frac{3}{8}$ (solid curve), and $\varepsilon = \frac{3}{4}$ (dashed curve). The homogeneous solution ($\varepsilon = 0$) is shown in dotted ($\xi_0 = 0$, $\ell = 10$, $E_1 = 1$, $E_2 = \frac{1}{2}$, $\rho_1 = 0.8$, $\rho_2 = 1$, $\alpha = 0.2$, $T = 1$, and the wave speed c is given by (37)).

4.3. Second-order correction

The $O(\varepsilon^2)$ system yields

$$x_{2,s} = -\tfrac{1}{2}\phi_1^2 \cos\phi_0 - \phi_2 \sin\phi_0, \tag{84a}$$

$$y_{2,s} = -\tfrac{1}{2}\phi_1^2 \sin\phi_0 + \phi_2 \cos\phi_0. \tag{84b}$$

While the leading-order and first-order systems are solved analytically above, the solutions we obtained for the second-order system are numerical. Integrating the coordinates x_2 and y_2 from (84) yields solutions depicted in Fig. 8(a) and (b) The arc length of the loop is altered by the second correction: note the shift in the x-coordinate versus the arc length variable ξ in Fig. 8(a). In the horizontal direction, the loop gets stretched out (cf. Fig. 8(a)), while in the vertical direction, the tails get slightly pushed upwards, while the central part of the loop is significantly pulled downwards (cf. Fig. 8(b)). The effect of the second correction on the overall shape of the loop is shown in Fig. 9 for different values of ε (compare with corresponding values of ε in Fig. 7).

5. Conclusion

We have examined planar loop traveling wave solutions on a heterogeneous, inextensible and unshearable elastic rod using a multiple scales homogenization technique, where the heterogeneity consisted in a periodic microstructure of two different material properties.

The leading-order balance (30) is a system describing a homogeneous rod, and allowed us to identify the effective material properties. In the case of the density ρ, it is just the bulk density $\rho_h = \langle\rho\rangle$, but as far as the Young modulus is concerned, the effective value is found to be $E_h = \langle E^{-1}\rangle^{-1}$.

Two lowest-order corrections have been found and shown to distort the homogeneous-type loop, breaking its symmetry. The effect of the corrections is shown in Fig. 9. From the effect of the second-order boundary conditions, we can infer that an initially traveling loop on a rod with periodic microstructure would eventually deform and loose its traveling wave structure as expected due to dispersion effects.

Acknowledgments

This publication is based in part upon work supported by the National Science Foundation under Grant DMS-0907773 and by Award no. KUK-C1-013-04, made by King Abdullah University of Science and Technology (KAUST) (A.G.). A.G. is a Wolfson/Royal Society Merit Award holder.

References

[1] M. Nizette, A. Goriely, Towards a classification of Euler–Kirchhoff filaments, J. Math. Phys. 40 (6) (1999) 2830–2866.
[2] B. Đuričković, A. Goriely, G. Saccomandi, Compact waves on planar elastic rods, Int. J. Non-Linear Mech. 44 (5) (2009) 538–544. doi:10.1016/j.ijnonlinmec.2008.10.007.
[3] J. Fish, W. Chen, Higher-order homogenization of initial/boundary-value problem, J. Eng. Mech. 127 (12) (2001) 1223–1230. doi:10.1061/(ASCE)0733-9399(2001)127:12(1223).
[4] T. McMillen, A. Goriely, Whip waves, Physica D 184 (2003) 192–225. doi:10.1016/S0167-2789(03)00221-5.
[5] J. Kevorkian, J.D. Cole, Multiple Scale and Singular Perturbation Methods, Springer, 1996.
[6] W. Chen, J. Fish, A dispersive model for wave propagation in periodic heterogeneous media based on homogenization with multiple spatial and temporal scales, J. Appl. Mech. 68 (2) (2001) 153. doi:10.1115/1.1357165.